PHYSNET

MISN-0-489

CONFORMAL MAPPING

Math Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

CONFORMAL MAPPING

by

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Title: **Conformal Mapping**

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Length: 2 hr; 8 pages

Input Skills:

1. Vocabulary: analytic functions, complex variable functions, contour integral, Laurent series, singularity, residue, simply connected region, Cauchy theorem, singular points, residues.
2. Unknown: assume (MISN-0-488).

Output Skills (Knowledge):

K1. Write the definition or explanation of each of the following terms and concepts: mapping and transformation, types of transformations: translation, rotation, stretching, inversion, linear, bilinear of fractional, conformal mapping, area magnification factor, linear magnification factor.

Output Skills (Rule Application):

R1. Determine the region in the w-plane into which a given region in the z-plane is mapped by an analytic function $f(z)$.

R2. Solve the boundary value problem involving Laplace's equation and Dirichet or Neumann B.C. by using conformal mapping techniques.

External Resources (Required):

1. G. Arfken, Mathematical Methods for Physicist, Academic Press (1995).
2. Schaum's Outline: Murray Spiegel, *Theory and Problems of Advanced Mathematics for Scientists and Engineers*, McGraw-Hill Book Co. (1971).

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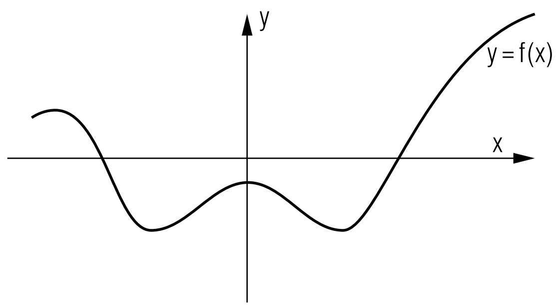
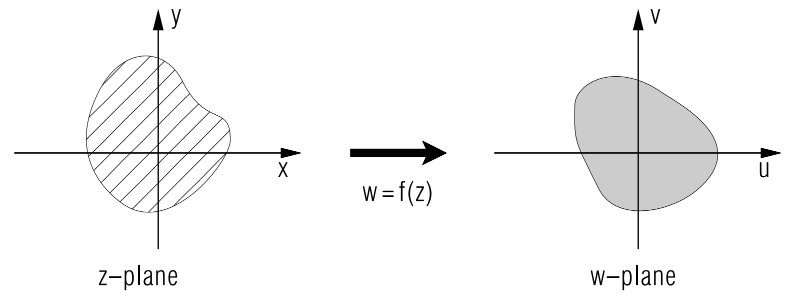
1. Introduction

In the previous units on complex variables, you have studied some of the most important properties of analytic functions from an algebraic or analytic point of view. In this unit you will study the geometric aspects of functions of a complex variable. These geometric aspects are important for understanding the various differential and integral operations involving analytic functions. To see why consider an ordinary real - valued function of a single, real variable, $y = f(x)$. Geometrically, this function can be displayed on a two-dimensional $x - y$ plane.



The derivative of $f(x)$ is then interpreted as the slope of the tangent to the curve at x while the integral of $f(x)$ from $x = a$ to $x = b$ is ,just the area inclosed by the curve and the x-axis between $x = a$ and $x = b$. If you wish to display a complex function of a complex variab]e, however, you will run into some difficulties because the complex variable $z = x + \imath y$ is two-dimensional as is the function $f(z) = u(x, y) + \imath v(x, y)$. Thus, you would need a four- dimensional space in order to display this function.

An alternative approach to inventing a four-dimensional space is to view the complex function geometrically as a mapping from one two-dimensional space (the z-plane) into another two- dimensional space [the w-plane where $w = f(z)$].



The transformation equations of this mapping are

$$u = u(x, y)$$

$$v = v(x, y)\,.$$

Thus, you can see how lines and areas in the z-plane map into lines and areas in the w-plane. If the function $f(z)$ is analytic, the mapping is called conformal and has some very special and important properties. These properties will allow you to solve some very difficult boundary value problems involving Laplace's equation in two dimens ions with surprising ease.

2. Procedures

1. Read these sections and pages in Arfken:

    Sec. 6.6, Mapping.

    Sec. 6.7, Conformal Mapping.

2. Read these sections and pages of Spiegel:

    Conformal Mapping, page 291

    Reimann's Mapping Theorem, pp 291 to 292

    Some General Transformations, p. 292

    Mapping of a Half Plane onto a Circle, pp 292 to 293

Solutions of Laplace's Equation by Conformal Mapping, p. 293

3. Underline in the texts or write out the definitions and explanations of the terms and concepts of Output Skill K1.

4. Read these Solved Problems in Spiegel:

   13.38 (Conformal Mapping Theory)

   13.39 (Conformal Mapping Theory)

   13.41 (Mapping upper half plane into unit circle by a bilinear transformation)

   13.46 (Harmonic Functions)

   13.47 (Harmonic Functions)

   13.48 (Laplace's Equation and Conformal Mappings)

   13.49 (Laplace's Equation and Conformal Mappings)

   13.50 (Harmonic Functions)

   13.51 (Solutions to Laplace's Equation using Conformal Mapping)

   13.52 (Solutions to Laplace's Equation using Conformal Mapping)

   13.53 (Solutions to Laplace's Equation using Conformal Mapping)*

   13.54 (Solutions to Laplace's Equation using Conformal Mapping)

   * Eq. 2 and 3 are incorrect. The correct equations are obtained by simply replacing the argument of the arc tangent by its reciprocal.

5. Here are some references that can be consulted for examples of mappings in the complex plane:

   a. Collins, *Mathematical Methods for Physicists and Engineers*, Reinhold (1968), Table 7-1, pages 158-159.

   b. Spiegel, *Theory and Problems of Complex Variables*, Schaum's Outline, McGraw-Hill, pages 205-211.

   The student should verify several of these mappings.

6. Solve these problems in Arfken:

   6.6.1 (Selected Mappings)

   6.6.2 (Selected Mappings)

   6.6.3 (c) (Selected Mapping $w = \sinh z$)

7. Solve these problems in Spiegel:

   13.110 (Selected Mappings)[1]

   13.111 (Selected Mappings)[2]

   13.112 (Selected Mappings)

   13.121 (Solving B. V. P. using Conformal Mapping)[3]

   13.122 (Solving B. V. P. using Conformal Mapping)[4]

Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

[1]In part (a), let $\mathrm{Im}\{z\} > 0$.
[2]The answers are incorrect: a) $u^2 - 2v = 1$, b) $u^2 + v^2 = u - v$.
[3]The correct answer is 85.6 °C.
[4]Remember that the answer to Solved Problem 13.53 of Spiegel is not correct. See the comment above.