PHYSNET

MISN-0-388

FORMAL STRUCTURE OF QM (I)

Quantum Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FORMAL STRUCTURE OF QM (I)

by

R. Spital

Title: **Formal Structure of QM (I)**

Author: R. Spital, Dept. of Physics, Illinois State Univ

Version: 2/1/2000 Evaluation: Stage B0

Length: 2 hr; 9 pages

Input Skills:

1. Unknown: assume (MISN-0-387).

Output Skills (Knowledge):

K1. State the superposition principle.

K2. Define "scalar product" and show that the result does not depend on whether the coordinate or momentum space representation of the wave-function is used.

K3. State: (1) the fundamental properties shared by all observable operators; and (2) the connection between the eigenvalues of an observable operator and measurement values of that observable.

K4. Define: hermitian operator and adjoint or hermitian conjugate.

K5. State the relationship between the commutator of two operators and one's ability to simultaneously measure the two observables to which they correspond.

K6. Define degenerate eigenvalue and degenerate eigenstates.

K7. Prove that the eigenvalues of a hermitian operator are real and the eigenfunctions of a hermitian operator are orthogonal if they correspond to distinct eigenvalues.

K8. Prove that if two operators commute and one has non-degenerate eigenvalues, its eigenfunctions are also eigenfunctions of the other operator.

Output Skills (Problem Solving):

S1. Given an expansion of the state of the system in terms of eigenvalues of the operator Q, calculate the probability of the various possible outcomes of a measurement of Q.

S2. Solve problems such as 6.1 through 6.5.

External Resources (Required):

1. E. E. Anderson, *Modern Physics and Quantum Mechanics*, W. B. Saunders (1971).

THIS IS A DEVELOPMENTAL-STAGE PUBLICATION
OF PROJECT PHYSNET

The goal of our project is to assist a network of educators and scientists in transferring physics from one person to another. We support manuscript processing and distribution, along with communication and information systems. We also work with employers to identify basic scientific skills as well as physics topics that are needed in science and technology. A number of our publications are aimed at assisting users in acquiring such skills.

Our publications are designed: (i) to be updated quickly in response to field tests and new scientific developments; (ii) to be used in both classroom and professional settings; (iii) to show the prerequisite dependencies existing among the various chunks of physics knowledge and skill, as a guide both to mental organization and to use of the materials; and (iv) to be adapted quickly to specific user needs ranging from single-skill instruction to complete custom textbooks.

New authors, reviewers and field testers are welcome.

FORMAL STRUCTURE OF QM (I)

by

R. Spital

1. Introduction

In this and the following unit we shall briefly explore the Hilbert Space in which the operators of quantum mechnaics merrily change one state into another. Our aim is to get a better understanding of the relationship between the various possible states of the system and the possible properties that these states can have.

2. Procedures

1. Read chapter 6 up to postulate 3 on page 206. The superposition principle is contained in postulate 2 and the subsequent discussion. Note that the c_i in equation 6.3 are in general *complex* numbers.

2. The *Scalar product* of two wave-functions ψ_a and ψ_b is defined by

$$< \psi_a|\psi_b >\equiv \int \psi_a^*(\vec{r})\, \psi_b(\vec{r})\, d^3\vec{r}$$

   Let ϕ_a and ϕ_b be the momemtum space wave-functions corresponding to ψ_a and ψ_b. Show that

$$< \phi_a|\phi_b >\equiv \int \phi_a^*(\vec{p})\, \phi_b(\vec{p})\, d^3\vec{p} =< \psi_a|\psi_b >$$

   Because the scalar product is *independent* of the *representation of the states* and depends *only* on the *states themselves*, we may write simply $< a|b >$ for the scalar product. How is $< a|b >$ related to $< b|a >$?

3. Continue reading through the beginning of the paragraph in which equation 6.12 appears. The necessary properties and the connection are contained in postulate 4 and the definitions following it.

4. The best deinition of "hermitian operator" to remember is equation 6.10. If you use equation 6.8, you must be prepared to derive equation 6.10 from it.

   The *adjoint* or *hemitian conjugate*, Q^+, of an operator Q (called "Hermitian adjoint" by the book) is defined by equation 6.11. Hermitian operators are therefore their own hermitian conjugates and are said to be "self-adjoint."

5. Read up to the theorem at the bottom of page 209 The discussion given applies to the case when Q is the Hamiltonion. More generally let $\psi = \Sigma_i c_i \psi_i$ where $Q\psi_i = q_i\psi_i$, i.e. the ψ_i are eigentates of Q with eigenvalues q_i. All possible eigenstates are present in the sum (although perhaps with a zero coefficient), and we assume for simplicity that the eigenstates are *non-degenerate*; i.e. there is a 1 to 1 correspondence between the eigenstates and the eigenvalues. The probability of obtaining q_i from a measurement of Q when the system is in the state ψ is $| < \psi_i|Q\psi > |^2$. Assume that $< \psi_i|\psi_j >= \delta ij$. (You will prove this below). Then show that $| < \psi_i|Q\psi > |^2 = |c_i|^2$ provided ψ is normalized.

   The *act of measurement* forces the system into an eigenstate of Q. If Q commutes with the Hamiltonian (see below), the system will remain in that eigenstate until disturbed again. We shall discuss this further in the next unit.

6. Read the 2 proofs and be sure you can reproduce them.

7. Read the rest of the section. Make sure you can derive equation 6.16. You need not prove the corollary at the bottom of page 211, equation 6.16 already gives the result. Now solve problems 6.1; 6.2 a, c; 6.3, 6.4, 6.5 to gain some practice with the ideas we've introduced. Once again, you are reminded that the energy operator is $\mathcal{H}$, and *not* the expression suggested in problem 6.2b.

8. Read section 2 through the end of the proof of the corollary on page 213. The conclusion that $[P, Q] = 0$ does not follow from the argument given in the book. It is *indeed* possible for ψ to be a simultaneous eigenstate of P and Q even if $[Q, P] \neq 0$. However, if for *every* eigenvalue of P, say p_i, there is an eigenfunction ψ_i which is also an eignefunction of Q, it follows that $[P, Q] = 0$.

   What are the implications of this for the measurement process? Problem 6-5 shows that unless ψ is an eigenfunction of Q, ΔQ for that state is non-zero. Therefore, in order to be able to measure P, Q simultaneously to arbitrary accuracy ($\Delta P = \Delta Q = 0$), we require the existence of a complete family of simultaneous eigenstates of P and Q. (More on "completeness" in the next unit). This in turn requires $[P, Q] = 0$. To summarize:

It is possible to simultaneously measure two observables to arbitrary accuracy if and only if they commute.

Commuting observables are called "compatible" for this reason. In view of the above discussion we see at once that we cannot simultaneously measure x and p_x to arbitrary accuracy, agreeing with the uncertainty principle. We also see that in order to have a set of energy eigenstates which are eigenstates of an operator Q, we require $[Q, \mathcal{H}] = 0$.

9. An eigenvalue is said to be *degenerate* if and only if there exist two linearly independent wave-functions ψ_1 and ψ_2 which are both eigenfunctions corresponding to the eigenvalue. (Linearly independent means that ψ_1 is not constant multiple of ψ_2.) Two eigenstates (eigenfunctions) are said to be degenerate if and only if they correspond to the same eigenvalue and are linearly independent.

10. Be certain that you can reproduce the proof of the corollary on page 213.

11. Read the rest of section 6.2. Be sure you can derive equation 6.18. Most operators do not explicity depend on the time so that $\partial Q/\partial t = 0$. Equation 6.18 then shows that the expectation value of Q is constant in time if $[Q, \mathcal{H}] = 0$. When a measurment of Q is made on a system, the system is forced into an eigenstate Of Q. The expectation value of Q in the eigenstate is, of course, the eigenvalue. If $[Q, \mathcal{H}] = 0$, this expectation value is constant in time. Let the eigenvalue (expectation value) be q_i. Referring to equation 6.16, what are the values of the c's as functions of time? How does this prove the statement at the end of procedure 4?

12. For additional practice, solve problems 6.6, 6.7 and 6.9.

Acknowledgments

The author would like to thank Illinois State University for support in the construction of this lesson. Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.