PHYSNET

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FOURIER ANALYSIS AND WAVE PACKETS

Quantum Physics

Project PHYSNET Physics Bldg. Michigan State University East Lansing, MI

FOURIER ANALYSIS AND WAVE PACKETS

by

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Title: **Fourier Analysis and Wave Packets**

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Input Skills:

1. Be able to integrate Gaussian function, exponential function, trigonometric function.
2. Write down the integral that produces the Fourier transform of an arbitrary fuction and the integral that transforms the Fourier transform back to the original function (MISN-0-380).

Output Skills (Rule Application):

R1. Use Fourier transformation to find the frequency (wavelength) spectrum of wave packets such as chopped sine and plane waves, and obtain the corresponding uncertainty relations.

Output Skills (Problem Solving):

S1. Calculate the Fourier transforms of functions such as chopped linear functions.

S2. Solve qualitative uncertainty principle problems.

External Resources (Required):

1. E. E. Anderson, *Modern Physics and Quantum Mechanics*, W. B. Saunders (1971).
2. D. S. Saxon, *Elementary Quantum Mechanics*, Holden-Day, Inc. (1968).

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1. Introduction

In the first two units we have seen the need to treat photons and particles as wave packets; i. e. as localizations of energy produced by the superposition of waves of different frequencies and wavelengths. In this unit we shall introduce the mathematical machinery needed to describe wave packets. This machinery is called Fourier analysis.

Armed with this new machinery, we shall then study a simple example of wave superposition in order to enhance our intuitive understanding of wave packets and the uncertainty principle.

2. Procedures

1. The readings for this unit are on reserve for you in the Physics-Astronomy Library. Ask for them as "the readings for CBI unit 383."

2. (Output Skill S1) Perform the necessary integrations to solve problem 3 on page 54 of Saxon.

3. a. (Output Skill R1) Read section 4.8 of Anderson through the middle of page 138. Verify the integrations done and make sure you can perform them without the aid of the book. Note that the uncertainty relations 4.31 and 4.32 are only qualitative; that is, the definition of $\Delta\omega$ or Δk is somewhat arbitrary. (There are more rigorous ways to define $\Delta\omega$ and Δk but we shall not take them up here.) The uncertainty products 4.31 and 4.32 are the minimum values achievable *at* $t = 0$. As time goes on, the packet will broaden in coordinate space so that these minimum values will increase. (The momentum space width Δp stays constant in time for a free particle. See problem 4.16).

   b. Read the remainder of section 4.8. (There is no difference between $\Phi(k)$ and $\phi(k)$ and ignore the incorrect end of the sentence at the top of page 139. Just end it after "normalize to a delta function.") $\phi(k)$ is the momentum space wavefunction. at $t = 0$. Using equation

4.33 and the definition $\Psi_k(x,t) = Ae^{i(kx-\omega t)}$ show that:

$$\phi(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,0)e^{-ikx}dx$$

In general:

$$\phi(k,t) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\Psi(x,t)e^{-ikx}dx$$

$\phi(k,t)$ is the time-dependent wave function in momentum space. For a free particle $\hbar\omega = p^2/2m$. Use this fact and equation 4.33 to find the function $\phi(k,t)$ in terms of $\phi(k) \equiv \phi(k,0)$. Then solve problem 4.16. Use an equation covered in an earlier Output Skill to solve problem 4.15. For additional practice with wave packets, solve problem 4.18 and add this question:

   c. $\Psi(x)$ is the wave packet at $t = 0$. Write an integral expression for $\Psi(x,t)$ in terms of $\Psi(x,0) \equiv \Psi(x)$. (Hint: Use the inverted form of the equation for $\phi(k,t)$ given above).

4. As an application of the uncertainty principle solve parts (a) and (b) of problem 4.19. Use the relation $\Delta p_i \Delta x_i \geq 1/2\hbar$ for each of the three components ($i = 1, 2, 3$) of the position and momentum vectors. The lower bound $\hbar/2$ is achieved by Gaussian wave packets and is the *absolute minimum* usually quoted in statements of the Heisenberg uncertainty principle.

Acknowledgments

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A. Solving Gaussian Integrals

(only for those interested)

The function e^{-ax^2} is-called a *Gaussian* function. It is frequently met in physics and in statistics so the method of solution of its integral is worthwhile knowing.

Method 1

Find the appropriate form in a Table of Integrals.[1] For example, in *A*

[1] We recommend that you own a table of integrals, such as that above, or: *Table of Integrals, Series and Products*, I. S. Gradshteyn and I. M. Ryzhik, Academic Press, New

Short Table of Integrals, B. O. Peirce, Ginn & Co. (Xerox Corporation), Boston (1929), one finds on page 63:

$$\text{491.}\quad \int_0 \sqrt{x} \qquad \int_0 \sqrt{\pi}$$

$$\text{492.}\quad \int_0^\infty e^{-a^2x^2}dx = \frac{1}{2a}\sqrt{\pi} = \frac{1}{2a}\Gamma(\tfrac{1}{2}). \quad a>0$$

$$\text{493.}\quad \int_0^\infty x^n e^{-ax}dx = \frac{\Gamma(n+1)}{a^{n+1}} . n>-1 \mathcal{F}\ a>0$$

$$\text{494.}\quad \int_0^\infty x^{2n}e^{-ax^2}dx = \frac{1\cdot 3\cdot 5\cdots(2n-1)}{2^{n+1}a^n}\sqrt{\frac{\pi}{a}} .$$

$$\text{495.}\quad \int_0^\infty e^{-x^2-\frac{a^2}{x^2}}dx \quad e^{-2a}\sqrt{\pi}$$

Hence:

$$\overline{x^2} = \sqrt{a/\pi}\int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,dx.$$

Because the integrand is symmetric, this can be written:

$$\overline{x^2} = \sqrt{a/\pi}\,2\int_0^{+\infty} x^2 e^{-ax^2}\,dx.$$

Now by No. 494 in Peirce's book this becomes:

$$\overline{x^2} = \sqrt{a/\pi}\;2\,\frac{1}{4a}\sqrt{\pi/a} = \frac{1}{2a}.$$

Method 2

Start with the square root of the square of the complete integral:

$$\begin{aligned}\int_{-\infty}^{+\infty} e^{-ax^2}\,dx &= \sqrt{\int_{-\infty}^{+\infty} e^{-ax^2}\,dx\;\int_{-\infty}^{+\infty} e^{-ay^2}\,dy}\\ &= \sqrt{\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-a(x^2+y^2)}\,dx\,dy}\\ &= \sqrt{\int_0^{+\infty} e^{-ar^2}\,2\pi r\,dr}\,.\end{aligned}$$

York and London (1965); or *Mathematical Tables from the Handbook of Chemistry and Physics*, Charles Hougman, Chemical Rubber Publishing Co. (1931 and later dates).

where the planar area integration has been re-expressed in polar coordinates. Now let $z \equiv r^2$ so that $dz = 2rdr$:

$$\int_{-\infty}^{+\infty} e^{-ax^2}\,dx = \sqrt{\pi\int_0^{+\infty} e^{-az}\,dz} = \sqrt{\pi/a}.$$

$$\begin{aligned}\overline{x^2} &= \int_{-\infty}^{+\infty} x^2 e^{-ax^2}\,dx\\ &= -\frac{d}{da}\int_{-\infty}^{+\infty} e^{-ax^2}\,dx\\ &= -\frac{d}{da}\sqrt{\pi/a} = \frac{1}{2a}\sqrt{\pi/a}\,.\end{aligned}$$