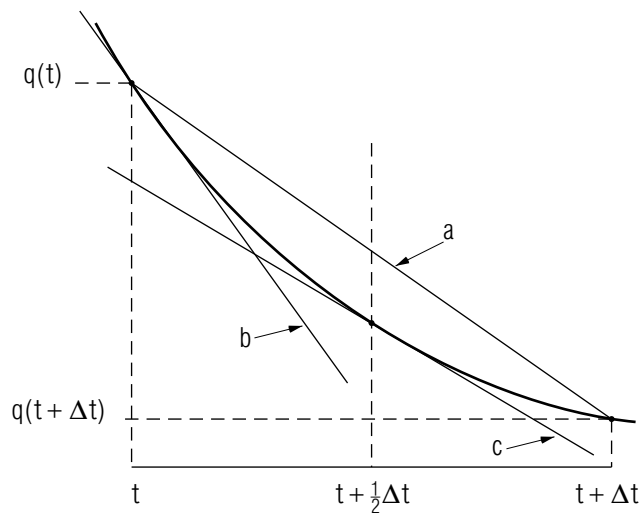


## EULER'S METHOD FOR COUPLED DIFFERENTIAL EQUATIONS; RLC CIRCUITS



## EULER'S METHOD FOR COUPLED DIFFERENTIAL EQUATIONS; RLC CIRCUITS by Robert Ehrlich

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Title: **Euler's Method for Coupled Differential Equations; RLC Circuits**

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Length: 2 hr; 20 pages

**Input Skills:**

1. Vocabulary: inductor (MISN-0-144).
2. Describe the Euler-method algorithm for solving first-order differential equations (MISN-0-350).

**Output Skills (Knowledge):**

- K1. Write an explanation of the improved Euler method for the numerical solution of two coupled first-order differential equations of the form  $dy_1/dx = f_1(y_1, y_2, x)$ ,  $dy_2/dx = f_2(y_1, y_2, x)$ .

**Output Skills (Project):**

- P1. Use the improved Euler method to solve the coupled, first order differential equations for an RLC circuit for free oscillations.
- P2. Use the improved Euler method to solve the coupled, first-order differential equations for an RLC circuit for oscillations that are forced by a sine-wave voltage generator.

**External Resources (Required):**

1. A computer with BASIC or FORTRAN or C++.

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# EULER'S METHOD FOR COUPLED DIFFERENTIAL EQUATIONS; RLC CIRCUITS

by  
Robert Ehrlich

## 1. Euler's Method

**1a. Original Euler's Method.** The simplest algorithm for the numerical solution of first-order differential equations of the form

$$\frac{dq}{dt}(t) = f[q(t), t], \quad (1)$$

where  $f$  is some known function, is based on the Euler method. This method replaces the differential equation by a finite difference equation through the substitution of<sup>1</sup>

$$\frac{dq}{dt}(t) \simeq \frac{\Delta q}{\Delta t} = \frac{q(t + \Delta t) - q(t)}{\Delta t} \quad (2)$$

into Eq. (1). This approximation is the basis of the original Euler method, wherein we solve Eq. (2) for  $q(t + \Delta t)$ :

$$q(t + \Delta t) = q(t) + \frac{dq}{dt}(t)\Delta t \quad (3)$$

and obtain  $dq/dt$  from Eq. (1).

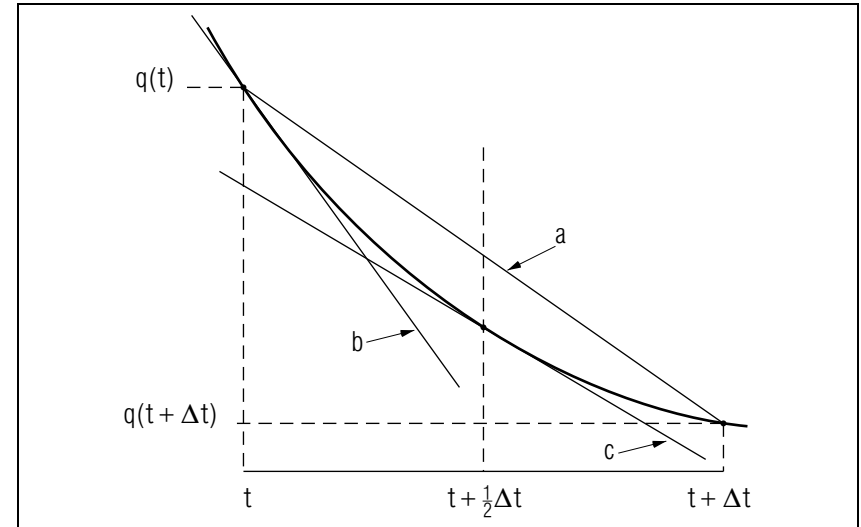
**1b. Improved Euler's Method .** The basis of the improved Euler's method is the observation that the right hand side of Eq. (2) is a better approximation to the derivative  $dq/dt$  at the intermediate time  $t + \Delta t/2$  than at the time  $t$ . This gives:

$$\frac{dq}{dt}(t + \frac{1}{2}\Delta t) = \frac{q(t + \Delta t) - q(t)}{\Delta t}. \quad (4)$$

The reason for this improvement is illustrated in Fig. 1. Equation (4) can be rearranged to give:

$$q(t + \Delta t) \simeq q(t) + \Delta t \frac{dq}{dt}(t + \frac{1}{2}\Delta t) \quad (5)$$

<sup>1</sup>See "Numerical Solutions of Differential Equations for RC Circuits Using Euler's Method" (MISN-0-350).



**Figure 1.** Graphical basis of the improved Euler method. The slope of curve (a) is the right side of both Eqs. (2) and (4), that of curve (b) is the left side of (2), and that of curve (c) is the left side of (4). It is obvious that (c)'s slope is a better approximation to (a)'s slope than is (b)'s.

where the derivative at the intermediate time,  $t + \Delta t/2$ , can be approximated by the average of the derivatives at times  $t$  and  $t + \Delta$ .

$$\frac{dq}{dt}(t + \frac{1}{2}\Delta t) \simeq \frac{1}{2} \left[ \frac{dq}{dt}(t + \Delta t) + \frac{dq}{dt}(t) \right]. \quad (6)$$

Equations (5) and (6) yield the improved Euler equation:

$$q(t + \Delta t) \simeq q(t) + \frac{\Delta t}{2} \left[ \frac{dq}{dt}(t) + \frac{dq}{dt}(t + \Delta t) \right]. \quad (7)$$

Equation (7) represents a substantial improvement on Euler's original method given in Eq. (3), but it requires knowledge of  $dq/dt$  at time  $t + \Delta t$  which, through Eq. (1), requires  $q(t + \Delta t)$  and that is the very quantity we are trying to find! The "improved Euler method" specifies that the required  $dq/dt$  at  $t + \Delta t$  is to be obtained by: (1) getting a crude estimate of  $q(t + \Delta t)$  from Eq. (3); and (2) putting that crude estimate into Eq. (1) to get the required derivative.

**1c. The RLC Circuit .** In the circuit shown in Fig. 2, an EMF (an applied voltage  $V_a$ ) is connected across the series combination of a switch  $S$ , a resistor  $R$ , an inductor  $L$ , and a capacitor  $C$ . We are interested in determining how the current in the circuit varies as a function of time after the switch is closed. Kirchhoff's second rule, otherwise known as the loop equation, states that around any closed loop the sum of the potential drops equals the sum of the EMF's. Applying this rule to the circuit shown in Fig. 2 after the switch is closed gives:

$$V_R + V_L + V_C = V_a . \quad (8)$$

The voltages across  $R$ ,  $L$ , and  $C$  can be expressed in terms of the current  $I$ , its time rate of change  $dI/dt$ , and the charge  $q$  on the capacitor  $C$ :

$$V_R = IR; V_L = L \frac{dI}{dt}; V_C = q/C . \quad (9)$$

Substitution of Eqs. (9) into (8) yields

$$L \frac{dI}{dt} + IR + q/C = V_a$$

or:

$$\frac{dI}{dt} = \frac{1}{L} [V_a - IR - q/C] . \quad (10)$$

In order to find the current  $I$  as a function of time, we must solve Eq. (10) together with the relation

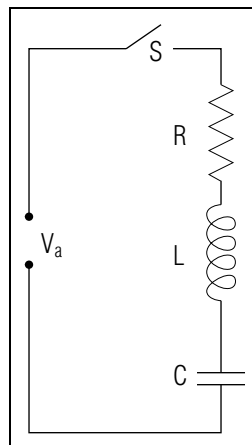
$$I = \frac{dq}{dt} , \quad (11)$$

which is the definition of the current flowing away from one plate of the capacitor. This is also the current which flows through the elements  $R$  and  $L$  because this is a single loop circuit. Equations (10) and (11) can be solved using the Euler methods. The two special cases we shall consider are:

- 1 Free oscillations of the  $RLC$  circuit when  $V_a = (\text{applied voltage}) = 0$ ;
- 2 Forced oscillations of the  $RLC$  circuit with  $V_a = V_0 \sin \omega t$ ;

The two first order differential equations, Eqs. (10) and (11), can be made into a single second order differential equation by differentiating Eq. (10) with respect to time and substituting it into Eq. (11). This gives:

$$L \frac{d^2 I}{dt^2} + R \frac{dI}{dt} + \frac{I}{C} + \frac{dV_a}{dt} = 0 . \quad (12)$$



**Figure 2.** The RLC series circuit.

Methods other than Euler's must be used to solve the problem in this form.<sup>2</sup>

**1d. Applying the Two Euler Methods to RLC Circuits.** In the original Euler method we use the first order Taylor expansions:

$$\begin{aligned} q(t + \Delta t) &= q(t) + q'(t) \Delta t \\ I(t + \Delta t) &= I(t) + I'(t) \Delta t \end{aligned} \quad (13)$$

where first derivatives have been indicated by primes. In the improved Euler method the derivatives  $q'(t)$  and  $I'(t)$  in Eq. (13) are replaced by the average values:

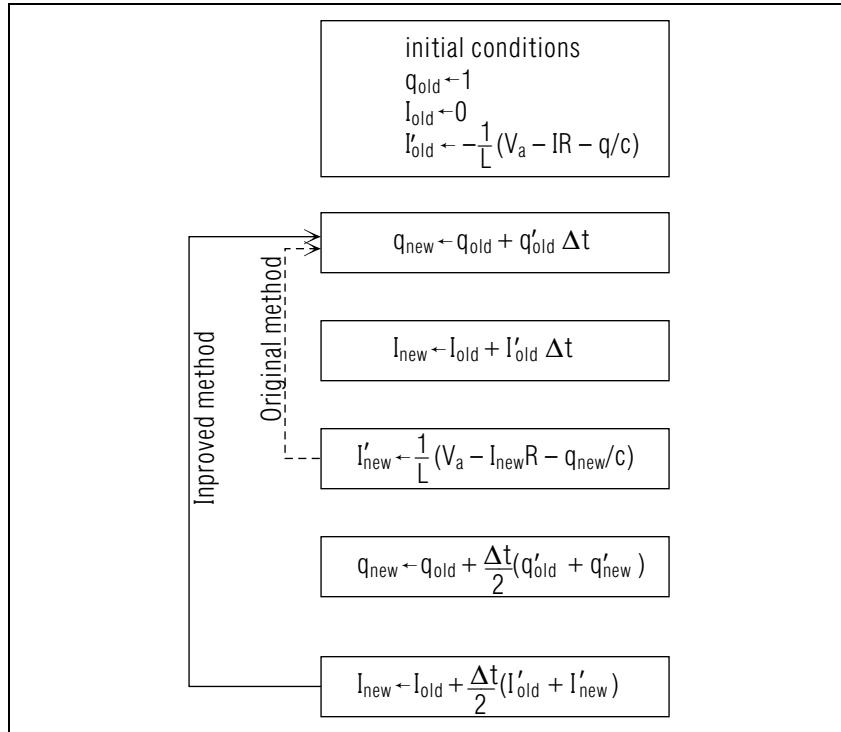
$$\frac{1}{2} [q'(t) + q'(t + \Delta t)]$$

and

$$\frac{1}{2} [I'(t) + I'(t + \Delta t)] .$$

Note that:  $I(t) \equiv q'(t) \equiv dq/dt$ . The exact sequences of steps followed in the two Euler methods are shown in Fig. 3. In order to begin the process we must specify two initial conditions, since there are two first order differential equations (or one second order equation). We shall arbitrarily assume the initial conditions,  $I = 0$  and  $q = 1$  at time  $t = 0$ . We next calculate values for the charge and current at the next value of the time,

<sup>2</sup>See "The Runge-Kutta Method for Solving Differential Equations," (MISN-0-367) or "The Numerov Method for Solving Differential Equations, Illustrated with the Damped Driven Oscillator," (MISN-0-39).



**Figure 3.** Flow diagram for the two Euler methods.

$t + \Delta t$ , using the original Euler's method [Eq. (13)], as well as an updated value for the derivative  $I'(t + \Delta t)$ . For the original Euler's method we advance the time by  $\Delta t$  ( $t \rightarrow t + \Delta t$ ), and loop back to repeat the process (dotted line). In the improved Euler method we compute improved values for  $q(t + \Delta t)$  and  $I(t + \Delta t)$  before completing the loop. Note that it was necessary to compute the original values for  $q(t + \Delta t)$  and  $I(t + \Delta t)$  before we could get the improved values since the improved values make use of the derivatives,  $q'(t + \Delta t) \equiv I(t + \Delta t)$  and  $I'(t + \Delta t)$ , found from Eq. (10).

## 2. Several Exact Solutions

**2a. Zero Applied Voltage .** If the applied voltage  $V_a$  is zero, then in certain cases free oscillations can occur in the  $RLC$  series circuit. If  $V_a$  is taken to be zero for all times, then  $dV_a/dt$  is also zero, in which case

Eq. (12) has the exact solution:

$$I = I_0 e^{-(R/2L)t} \sin \omega_0 t, \quad (14)$$

where

$$\omega_0 = \frac{1}{\sqrt{LC}} \left(1 - \frac{R^2 C}{4L}\right)^{1/2}. \quad (15)$$

The computer-generated curve of Fig. 4 is the approximate solution obtained using the improved Euler's method. The gradually decaying oscillations of the curve agree with the general form of the exact solution, in which an oscillatory factor,  $\sin \omega_0 t$ , is multiplied by a "damping factor"  $\exp[-(R/2L)t]$ . As long as  $R^2 C/4L$  is much less than one, then Eq. (15) which gives a value for  $\omega_0$ , reduces to

$$\omega_0 \approx \frac{1}{\sqrt{LC}}; \quad \left[\text{for } R \ll (4L/C)^{1/2}\right]; \quad (16)$$

where  $\omega_0$  is  $2\pi$  times the frequency of the free oscillations. The period  $T$  of the oscillations is given by:

$$T = \frac{2\pi}{\omega_0} \approx 2\pi\sqrt{LC}. \quad (17)$$

The appearance of the damping factor  $e^{-(R/2L)t}$ , in Eq. (14), implies that the envelope i.e., the non-oscillatory curve drawn tangent to the solution (see the hand-drawn dotted curve in Fig. 4), should fall to a fraction  $1/e \approx 0.368$  of its initial value after a time

$$\tau = \frac{2L}{R}. \quad (18)$$

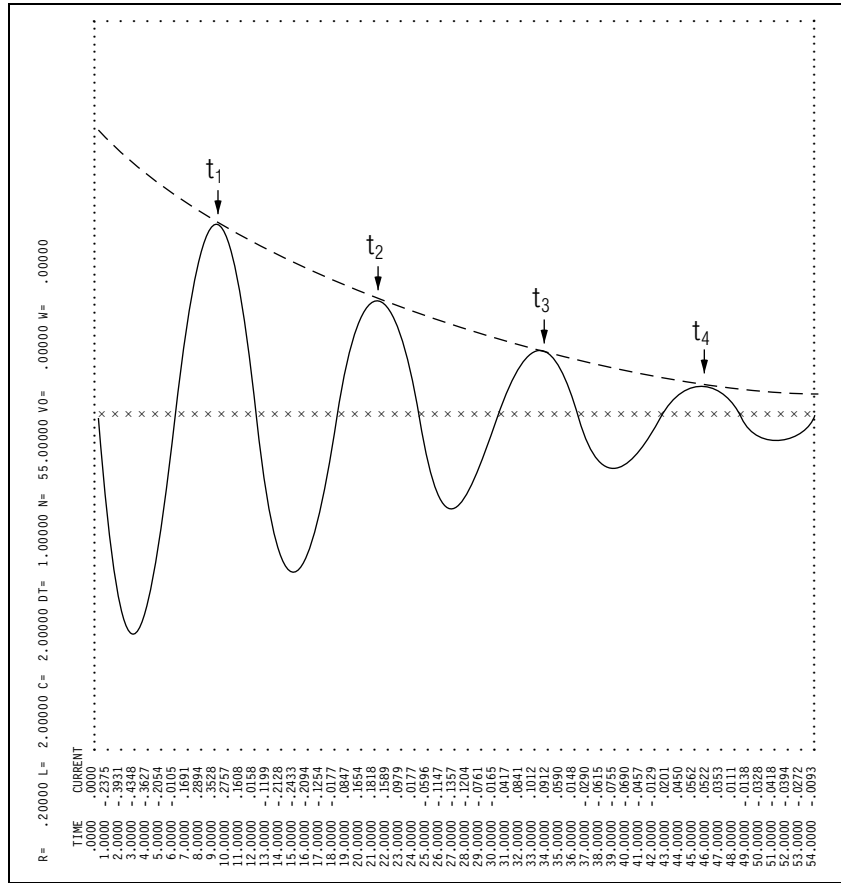
By examining the expressions for  $\omega_0$  in Eq. (16) we see that  $\omega_0$  will be a real number only if

$$1 - \frac{R^2 C}{4L} > 0,$$

which therefore requires that:

$$R < (4L/C)^{1/2}. \quad (19)$$

If  $R$  exceeds the critical value  $\sqrt{4L/C}$ , then  $\omega_0$  becomes imaginary and the factor  $\sin \omega_0 t$  is replaced by the exponential of a real number. For this "overdamped" case the solution becomes a pure decaying exponential and no longer oscillates. For this reason, when  $R$  has the borderline value,  $R = \sqrt{4L/C}$ , the circuit is said to be critically damped.



**Figure 4.** Computer output for the free underdamped oscillator (hand-drawn solid line). Also shown is the exponential damping factor (dashed line), computed directly from its exact formal expression. The times  $t_1, \dots, t_4$  are one period apart.

**2b. Sinusoidal Applied Voltage.** If the applied voltage is a sinusoidal function of time, then forced oscillations occur in the  $RLC$  circuit. When the applied voltage is given by  $V_a = V_0 \sin \omega t$ , then Eq. (12) again has an exact solution. If the initial conditions are chosen properly, the current is a sinusoidal function of time, with the same oscillation frequency

as the applied voltage:

$$I = I_m \sin(\omega t - \phi), \quad (20)$$

$$I_m = \frac{V_0}{Z} \quad (21)$$

and the quantity  $Z$  is known as the impedance and is given by

$$Z = \left[ R^2 + \left( \omega L - \frac{1}{\omega C} \right)^2 \right]^{1/2}. \quad (22)$$

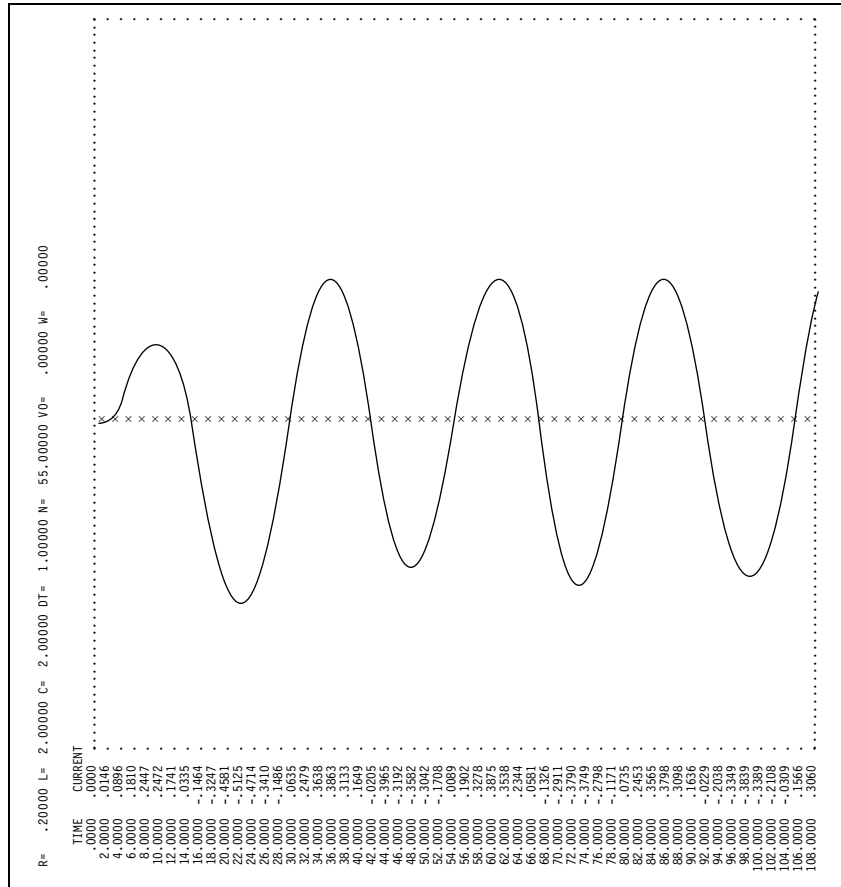
The quantity  $\phi$  appearing in Eq. (20) is the phase shift which is the relative shift between the sine curves describing the applied voltage and the current as functions of time;  $\phi$  is given by

$$\phi = \tan^{-1} \frac{\left( \omega L - \frac{1}{\omega C} \right)}{R}. \quad (23)$$

If the initial conditions are chosen arbitrarily, the solution will not have a purely sinusoidal time dependence. Generally these departures from a sinusoidal shape occur at times close to  $t = 0$  and then gradually disappear. For this reason such initial departures from a sinusoidal shape are known as transient currents. They occur whenever the initial conditions are other than those consistent with the specific solution which is purely sinusoidal [Eq. (20)]. Figure 5 shows a computer generated solution obtained using the improved Euler's method. The initial conditions used in this case ( $q_0 = 1$  and  $I_0 = 0$ ), are such that there evidently is a transient current present initially. After the transient current dies out, the amplitude and phase of the resulting sine wave should be consistent with that given by Eq. (21).

**2c. Resonance in  $RLC$  Circuit .** Like any system whose oscillations are driven by an external source, an  $RLC$  circuit can resonate when the oscillation frequency of the external source matches the "natural" frequency of the circuit. (The natural frequency is the frequency of free oscillations). At resonance the amplitude of the current has a greater value than for any other frequency. To find the frequency at which the amplitude of the current  $I_m = V_0/Z$  reaches its maximum, we must find the frequency for which  $Z$  is a minimum. Since the factor  $(\omega L - 1/\omega C)^2$  which appears in Eq. (22) can not be negative,  $Z$  will have its smallest possible value when this factor is zero:

$$\left( \omega L - \frac{1}{\omega C} \right)^2 = 0. \quad (24)$$



**Figure 5.** Computer output for  $V_a = V_0 \sin \omega t$ , showing transient current present at times near  $t = 0$ . At much later times the function becomes purely sinusoidal.

Solving Eq. (24) for  $\omega$  yields

$$\omega = \frac{1}{\sqrt{LC}}. \quad (25)$$

Note that the resonant frequency given by Eq. (25) is the same as the approximate frequency of free oscillations, given by Eq. (16). This means that the circuit shows the greatest “response” (maximum amplitude current), when the applied voltage has a frequency equal to the natural frequency, i.e., the frequency of free oscillations.

**2d. Error Analysis .** It is easy to compare the accuracy of the improved and original Euler method’s. In the improved Euler method we find  $q(t + \Delta t)$  using Eq. (7) in which the derivative  $dq/dt(t + \Delta t)$  needs to be evaluated using the original Euler method, that is:

$$\frac{dq}{dt}(t + \Delta t) = \frac{dq}{dt}(t) + \Delta t \frac{d^2q}{dt^2}(t) \quad (26)$$

Combining Eqs. (7) and (26) yields:

$$q(t + \Delta t) = q(t) + \Delta t \frac{dq}{dt}(t) + \frac{\Delta t^2}{2} \frac{d^2q}{dt^2}(t). \quad (27)$$

This is the correct Taylor expansion for  $q(t + \Delta t)$  up to terms second order in  $\Delta t$ . The size of the  $\Delta t^2$  term therefore, represents the difference between the original and improved Euler methods. Clearly, the accuracy of both methods improves as  $\Delta t$  decreases. In the case of the *RLC* circuit there are three time intervals of importance:

1. The period of free oscillations  $T = 2\pi/\omega_0$ , where  $\omega_0$  is given by Eq. (15).
2. The damping time for free oscillations,  $\tau = 2L/R$ .
3. The period of the forced oscillations  $T = 2\pi/\omega$ , where  $\omega$  is the frequency of the applied voltage:  $V_a(t) = V_0 \sin \omega t$ .

In order for either the original or the improved Euler method to give accurate results, the time step  $\Delta t$  must be chosen to be small compared to all three of these times.

### 3. Computer Implementation

**3a. Input.** The program<sup>3</sup> first reads numerical values for the parameters R, L, C, D, N, VO, and W, where

R	=	resistance
L	=	inductance
C	=	capacitance
DT	=	$\Delta t$ (time step)
N	=	number of time steps
VO	=	$V_0$ (amplitude of applied voltage)
W	=	$\omega$ (frequency of applied voltage).

<sup>3</sup>For BASIC, FORTRAN, and C++ implementations, see this module’s Computer Program Supplement.

After reading the data and setting the initial conditions, the program proceeds to advance the time in the sequence:  $0, \Delta t, 2\Delta t, 3\Delta t, \dots, N\Delta t$ . For each new value of the time, it calculates new values of the charge and current from the old values, using the improved Euler's method. In addition to printing out a table of numerical values giving the current at each time, the program also displays a graph of current against time.

**3b. Output.** The sample output shown in Figs. 4 and 5 were obtained using two data cards with the following numerical values for the parameters:

	R	L	C	DT	N	V0	W
Fig. 4	0.2	2.0	2.0	1.0	55.0	0.0	0.00
Fig. 5	1.0	8.0	8.0	2.0	55.0	0.7	0.25

In the first case, since the applied voltage is zero ( $V_0 = 0$ ), and since the values used for R, L, and C satisfy inequality (19), the solution is a damped oscillatory one, as previously discussed. In the second case, the sinusoidally varying applied voltage ( $V_0 \neq 0$ ), yields a solution which is also sinusoidal, apart from a transient effect.

## 4. A Project: Examine Accuracy

**4a. Accuracy For Free Oscillations.** Run the program using a number of data sets all of which have  $V_0 = 0$  and various values for the other parameters. Verify that the inequality (19) must be satisfied if the solution is to be an oscillatory one, by using values of R slightly less than and slightly greater than the value for critical damping. For the oscillatory solutions, see if their detailed shape agrees with the form of the exact solution (equation). To verify that the shape of the curve in Fig. 4 is correct, we may identify the times  $t_1, \dots, t_4$  at which the current  $I$  has its positive maxima: Reading Fig. 4, for example, we find:

	$t_1$	$t_2$	$t_3$	$t_4$
current ( $I$ )	0.2757	0.1818	0.0912	0.0522
time ( $t$ )	10.0000	21.0000	34.0000	46.0000

The time from  $t_1$  to  $t_4$  represents three periods, so we empirically find  $T = (t_4 - t_1)/3 = 12$  sec. The theoretical value for the period may be found using Eq. (15) which yields  $W = 0.494$  and  $T = 2\pi/\omega = 12.6$  sec. Carry out a similar analysis for your own data.

**4b. Accuracy For Forced Oscillations, Resonances.** Run the program using a number of data sets with various values for the parameters, except for V0 which is kept at some fixed nonzero value. In particular, try a range of values for W with the other parameters held fixed. See if the maximum amplitude current is observed at the predicted resonant frequency, [Eq. (25)]. Use enough values for W, above and below resonance, to be able to plot by hand a resonance curve (amplitude of the current versus frequency). Note, that when you vary W you should also vary DT, since  $DT = 0.5/W$ . This will have the effect of giving the same number of points per cycle for each run.

## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and Research, through Grant #SED 74-20088 to Michigan State University.

## A. Fortran, Basic, C++ Programs

All programs are at

[http://www.physnet.org/home/modules/support\\_programs](http://www.physnet.org/home/modules/support_programs)

which can be navigated to from the home page at

<http://www.physnet.org>

by following the links:  $\rightarrow$  modules  $\rightarrow$  support programs, where the programs are:

m351p1f.for, Fortran;  
 m351p1b.bas, Basic;  
 m351p1c.cpp, C++;  
 lib351.h, needed Library for C++ program;



## MODEL EXAM

1. Write an explanation of the improved Euler method for the numerical solution of two coupled first-order differential equations of the form  $dy_1/dx = f_1(y_1, y_2, x)$ ,  $dy_2/dx = f_2(y_1, y_2, x)$ .

**Examinee:**

On your computer output sheet(s):

- (i) Mark page numbers in the upper right corners of all sheets.
- (ii) Label all output, including all axes on all graphs.

On your Exam Answer Sheet(s), for each of the following parts of items (below this box), show:

- (i) a reference to your annotated output; and
- (ii) a blank area for grader comments.

When finished, staple together your sheets as usual, but include the original of your annotated output sheets just behind the Exam Answer Sheet.

2. Submit the graphical results of your free-oscillation runs using the improved Euler method, with your hand-annotated output demonstrating that:
- a. an oscillatory solution requires Eq. (19) when, done by graphically displaying the shapes found for  $R$  values slightly below and above the critical value;
  - b. the period of your oscillatory solution is correct, done by deducing the value of  $T$  from at least 3 periods and comparing to your computed theoretical value; and
  - c. the general shape of the solution is correct, done by plotting it.
3. Submit the graphical results of your forced-oscillation runs using the improved Euler method, with your hand-annotated output succinctly demonstrating that:
- a. your maximum current amplitude occurs at your formula-calculated resonant frequency (shown on a plot of frequency versus current amplitude).

- b. you properly used the same number of points per wave period for each run.

## INSTRUCTIONS TO GRADER

If the student has submitted copies rather than originals of the computer output, state that on the exam answer sheet and **immediately stop grading the exam and give it a grade of zero.**

Note that the award of points is set up in such a way that a student will get 50% or less on the exam if the student does only computer work and no physics.

