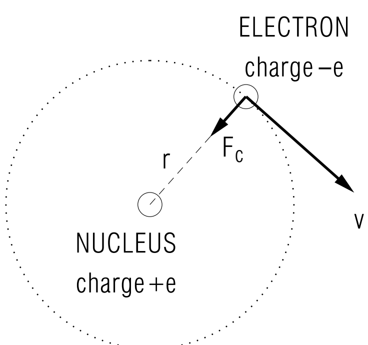
PHYSNET

MISN-0-309

# THE BOHR-SOMMERFELD MODEL OF THE ATOM

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by
Paul M. Parker

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Author: Paul M. Parker, Michigan State University

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**Input Skills**:

1. Vocabulary: energy level diagram, ground state, excited state, quantization, principal quantum number, ionization, hydrogen-like ion, atomic number (MISN-0-215); de Broglie wave (MISN-0-240), Avogadro's number (MISN-0-157).
2. Calculate the potential energy of a system of two charged particles (MISN-0-117).
3. Express a centripetal force in terms of the speed and orbital radius of a mass undergoing uniform circular motion (MISN-0-17).
4. State the angular momentum of a particle of mass m undergoing uniform circular motion (MISN-0-34).

**Output Skills (Knowledge)**:

K1. Vocabulary: Bohr radius, Rydberg constant.

K2. Derive the allowed radii, allowed speeds, and allowed energies of electrons in the Bohr model.

K3. State Sommerfeld's three refinements of the Bohr model.

K4. State the deficiencies and limitations of the Bohr-Sommerfeld model.

K5. State the differences between the quantum mechanical picture of the atom and the Bohr-Sommerfeld model.

**Output Skills (Problem Solving)**:

S1. Given the masses and charges of a two-particle atomic system, calculate the allowed energies of the system and draw an energy level diagram to scale.



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## 1. Overview

In this module Bohr's model as applied to the most common one-electron system, the hydrogen atom, is described first. The model is extended to other one-electron systems (the hydrogen-like ions) and chemical and spectroscopic evidence in support of the model is introduced. Finally, the limitations and deficiencies of the model are discussed.

## 2. Basic Features

**2a. Introduction.** The Danish physicist Niels Bohr proposed the first modern model of atomic structure in 1913. It applied only to the simplest atomic systems, namely, those consisting of an atomic nucleus plus a single orbiting electron. Although Bohr's model is now considered naive and outdated, some of its important features have survived unchanged, while others have persisted at least in a modified or qualitative manner. The model is, therefore, an excellent entry point to the study of atomic structure and spectroscopy.

**2b. Electrons Move in Circular Orbits.** Guided by earlier work by the nuclear physicist Ernest Rutherford, Bohr visualized the hydrogen

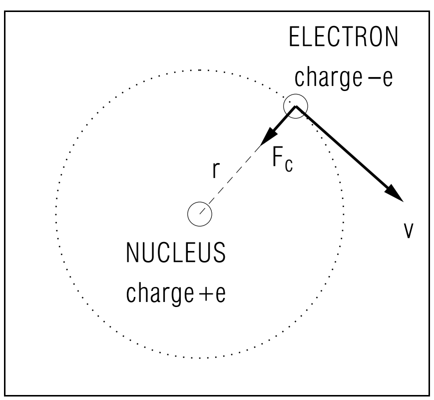


**Figure 1.** Uniform circular motion of electron about the nucleus.





atom as consisting of a massive, compact and positively charged nucleus and a negatively charged, point-like electron of mass m moving with constant speed v in a circular orbit around the nucleus (see Fig. 1). The centripetal force $\vec{F}$ required to keep the electron in a circular orbit is the electric force of attraction between the positive nucleus and the negative electron and by Coulomb's law has the magnitude $F_c = ke^2/r^2$ where $k = 8.988 \times 10^9\,\mathrm{N\,m^2/C^2}$. This force should satisfy Newton's law:

$$\vec{F}_c = m\vec{a}\,, \tag{1}$$

where $a = v^2/r$ is the electron's centripetal acceleration, thus:

$$\frac{ke^2}{r^2} = m\frac{v^2}{r}\,. \tag{2}$$

**2c. Angular Momentum is Quantized.** Additionally, Bohr made the radical and drastic assumption that the magnitude of the angular momentum of the electron, $L = mvr$, is not only constant, as it would be in Newtonian mechanics, but also quantized, that is, restricted to selected, "allowed" values governed by the positive integers and Planck's constant $h = 6.626 \times 10^{-34}\,\mathrm{J\,s}$, as follows:

$$L = mvr = n\hbar; \qquad n = 1, 2, 3, 4, \ldots\,, \tag{3}$$

where $\hbar$ (pronounced "h-bar") denotes $h/2\pi$. At the time, the only justification for Eq. (3) was that it made the model work. We now know that the quantization of angular momentum is a direct consequence of the electron's wave nature which was not discovered until more than a decade after Bohr's work. According to the wave model of the atom, Eq. (3) is incorrect. The quantity "$n$," now called the "principal quantum number," still has the same set of values although it arises in a totally different manner.[1]

## 3. Consequences of Quantized $L$

**3a. "Allowed" Orbital Radii.** We may eliminate $v$ from Eq. (2) and Eq. (3) and solve the resulting expression for $r$:

$$r_n = \frac{n^2\hbar^2}{kme^2}\,, \tag{4}$$

[1] See "The Schrödinger Equation in One Dimension: Quantization of Energy" (MISN-0-242).

where $r$ has been subscripted to emphasize that it depends on $n$ and is therefore quantized; that is, only selected values of $r$ are allowed. The smallest allowed orbit radius is obtained by taking $n = 1$,

$$r_1 = \frac{\hbar^2}{kme^2}\,. \tag{5}$$

By supplying the known numerical values of the fundamental constants $h$, $k$, $m$ and $e$, the radius of the smallest allowed Bohr orbit (called the "Bohr radius") can be calculated:

$$r_1 = 5.29 \times 10^{-11}\,\text{m} = 0.529\,\text{Å} = 0.0529\,\text{nm}\,, \tag{6}$$

and the $n^{\text{th}}$ radius is

$$r_n = n^2 r_1\,. \tag{7}$$

**3b. "Allowed" Orbital Speeds.** When we substitute the expression for $r_n$ back into Eq. (3) and solve the resulting equation for the allowed values of $v$, we obtain

$$v_n = v_1/n\,, \tag{8}$$

where $v_1$ is the speed of the electron in the first Bohr orbit,

$$v_1 = \frac{ke^2}{\hbar} = 2.19 \times 10^6\,\text{m/s} \approx \frac{c}{137}\,, \tag{9}$$

and $c$ is the speed of light in vacuum.

**3c. "Allowed" Electron Energies.** The total energy $E$ of the electron in one of its allowed orbits is the sum of its kinetic energy $E_k = mv^2/2$ and its potential energy $E_p = -ke^2/r$ in the electric field of the nucleus, with the conventional choice of $r = \infty$ corresponding to zero electric potential energy. Since the electrical force on the electron is attractive, moving the electron from $r = \infty$ to some finite distance $r$ from the nucleus lowers its potential energy below zero, that is, to negative values. The total energy is

$$E = \frac{1}{2}mv^2 - \frac{ke^2}{r}\,. \tag{10}$$

Using Eq. (2) we replace $(mv^2/2)$ by $(ke^2/2r)$, producing

$$E = -\frac{ke^2}{2r}\,. \tag{11}$$





Introducing into this expression the allowed values of $r$ from Eq. (4) gives the corresponding allowed energies,

$$E_n = -\frac{1}{n^2}\left(\frac{k^2me^4}{2\hbar^2}\right) = -\frac{R}{n^2}\,, \tag{12}$$

where $R$ is called the "Rydberg constant,"

$$R = \frac{k^2me^4}{2\hbar^2}\,. \tag{13}$$

Given the known numerical values of $k$, $m$, $e$ and $h$, the Rydberg constant is calculated to be

$$R = 2.18 \times 10^{-18}\,\text{J} = 13.6\,\text{eV}\,. \tag{14}$$

Table 1. The allowed energies of the electron in atomic hydrogen.

| $n$ | Energy |
|---|---|
| 1 | $E_1 = -13.6$ eV |
| 2 | $E_2 = -\ 3.40$ eV |
| 3 | $E_3 = -\ 1.51$ eV |
| 4 | $E_4 = -\ 0.850$ eV |
| . | |
| . | |
| . | |
| $\infty$ | $E_\infty = 0$ |

The allowed electron energies can be calculated from Eq. (12) as shown in Table 1 and marked off on a vertical energy axis to give the "energy level diagram" shown in Fig. 2. Since physical systems generally tend toward the lowest possible energy state, an isolated hydrogen atom will normally be found in the lowest allowed energy state ($n = 1$) which is called the "ground state." The Bohr model predicts that additionally the atom can exist in an infinite number of "excited states" with quantized energies $E_n = -R/n^2$, $1 < n < \infty$. In the limit as $n \to \infty$, both $E_k$ and $E_p$ become zero and this is taken to mean that the electron is no longer bound to the nucleus, i.e. the system has become "ionized." The minimum energy required to ionize the hydrogen atom is called the "ionization energy" and is 13.6 eV since this is the energy which must be supplied in some form to raise the atom from the ground state where

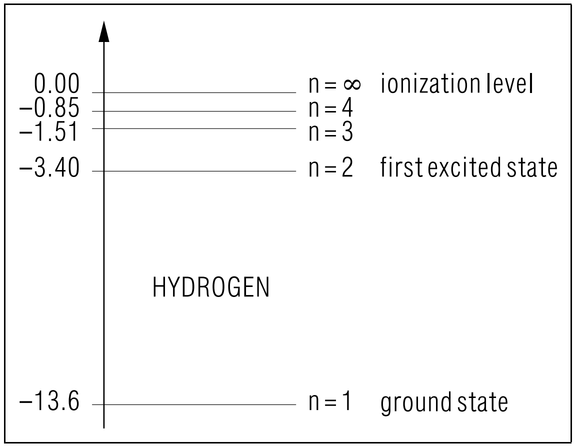


**Figure 2.** Energy level diagram of the allowed energies of the electron in atomic hydrogen.

it normally resides to the "ionization level." If more than this energy is supplied, the excess will reside with the liberated electron and the residual nucleus as kinetic energy.

## 4. Bohr Model for Other Systems

**4a. Overview.** The Bohr model can be applied to two-particle atomic systems other than atomic hydrogen. For example, it can be applied to any hydrogen-like ion that consists of a single electron orbiting a nucleus containing more than one proton. Such a system is obtained by ionizing (removing) all but one electron from an initially neutral atom. Other two-particle systems can be composed of exotic particles such as pions, muons, positrons, etc., instead of the usual electrons and protons. The limitations of the Bohr model are more apparent in some of these systems than in others.

**4b. Hydrogen-like Ions.** In order to rework the Bohr model for all hydrogen-like ions, the only modification to be taken into account is that whereas the nuclear charge for hydrogen was $+e$, for the hydrogen-like ion of element $Z$ it is $(+Ze)$. Coulomb's law, which for hydrogen was written as $F_c = ke^2/r^2$ then becomes $F_e = kZe^2/r^2$. Comparing the two expressions for $F_c$, it is seen that the hydrogen results can be adapted to the ions by simply replacing $(e^2)$ with $(Ze^2)$ wherever it occurs. This quick fix gives:

$$r_n = n^2 r_1/Z, \qquad v_n = Zv_1/n, \qquad E_n = -Z^2R/n^2\,, \tag{15}$$





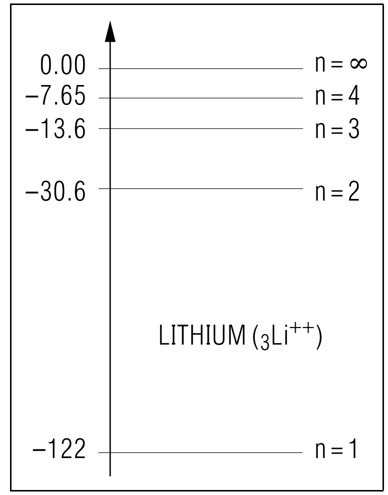


**Figure 3.** Energy level diagram for the hydrogen-like ion $_3\mathrm{Li}^{2+}$.

where, as before, $r_1 = 0.529\,\text{Å}$, $v_1 \approx c/137$ and $R = 13.6\,\text{eV}$. Taking $Z = 1$ in Eq. (15) reduces the expressions back to those for hydrogen. Because of the increased nuclear charge of the ion, the electron becomes more tightly bound than in hydrogen. The energies, therefore, become more negative, i.e., go deeper and become further removed from the ionization level, by a factor of $Z^2$. This is illustrated for $_3\mathrm{Li}^{++}$ in Fig. 3. As Z increases, the radii become smaller by the factor $Z$, and the speeds increase by the same factor, leaving the angular momenta unchanged as required by Eq. (3). These general trends are of considerable interest as one proceeds from element to element through the periodic table.[2]

**4c. Exotic Atomic Systems.** If an atomic system is constructed of exotic particles such as mesons, muons, positrons, etc., the energy and the orbital radius are scaled by the mass of the new system. For example, if we replace the electron in a hydrogen atom with a muon of mass $10.6\,\mathrm{MeV}/c^2$ and charge $-e$, the only change to the energy expression, Eq. (12), is to replace the mass of the electron with the mass of the muon. The result is:

$$E_n = -\frac{1}{n^2}\left(\frac{k^2 m_\mu e^4}{2\hbar^2}\right) = -\frac{1}{n^2}\left(\frac{m_\mu}{m}\right)\left(\frac{k^2 m e^4}{2\hbar^2}\right).$$

[2] See "The Pauli Principle and the Periodic Table of the Elements" (MISN-0-318).

Thus the energy levels are scaled up by a dimensionless factor equal to the ratio of the muon mass to the electron mass, i.e.,

$$\left(\frac{m_\mu}{m}\right) = \left(\frac{106\,\mathrm{MeV}/c^2}{0.511\,\mathrm{MeV}/c^2}\right).$$

The Bohr radii of such a "$\mu$-mesic atom" would similarly be scaled down by a factor of 207.

## 5. Supporting Experimental Evidence

**5a. The Size of the Atom.** Our first evidence supporting the Bohr picture is from the sizes of atoms. From the measured mass densities of solids and liquids, and with a knowledge of Avogadro's number, it is easy to infer that the radii of atoms are of the order of one angstrom. The radius of the first Bohr orbit of hydrogen is 0.529 Å, so it is in the right ballpark.

**5b. Hydrogen's Ionization Energy.** Next, the ionization energy of atomic hydrogen, according to the Bohr model, is 13.6 eV per atom. The molar ionization energy can be obtained from it by multiplication with Avogadro's number. After conversion to the caloric energy scale, the result is 22.4 kcal/mole which agrees with the value obtained through chemical heat of reaction measurements. The Bohr model gives this correct result solely as the specified combination of fundamental constants of nature. The odds against this being merely a "lucky coincidence" are overwhelming.

**5c. Optical Spectra of Hydrogen.** The most convincing and accurate evidence for the validity of the energy formula Eq. (12) comes from the optical spectra of hydrogen and the hydrogen-like ions. The observed frequencies of the spectra of hydrogen and the hydrogen-like ions agrees with the calculated values to four significant digits in all known cases. This is explored in detail elsewhere.[3]

## 6. Refinements of the Model

Three refinements of the Bohr model were worked out by Arnold Sommerfeld and are jointly responsible for giving rise to small deviations in the allowed energies from the values $E_n = -Z^2R/n^2$. These refinements brought the calculated frequencies into even better agreement with

[3] See "Energy Levels and Spectra of Atomic One-Electron Systems" (MISN-0-215).





observation (from four to five significant digits). The refinements were: (a) the introduction of elliptic orbits; (b) allowance for an orbiting motion of the nucleus; and (c) the consideration of relativistic mass effects. We discuss each of these in turn.

a. Elliptic orbits are useful in the interpretation of the shell and subshell structure of the atoms of the elements and the arrangement of the elements in the periodic table.

b. Allowing for nuclear motion, with the nucleus and electron both orbiting around their common center of mass, introduces a small dependence of the Rydberg constant on the mass of the nucleus. As a consequence, different mass isotopes of the same element have slightly different emission frequencies.[4] Deuterium, the heavier and rare stable isotope of hydrogen, was discovered in 1932 in the spectra of naturally occurring hydrogen samples and was later isolated in the laboratory.

c. Consideration of relativistic effects allowed Sommerfeld to calculate the breakup of the simple model's single frequency into the small number of closely spaced "fine structure" components observed with high quality optical equipment.

## 7. Limitations of the Model

Despite the satisfactory manner in which the Bohr-Sommerfeld approach accounts for the allowed energies and spectral frequencies of atomic one-electron systems, the model is seriously deficient and incomplete in several ways.

1. The assumption that angular momentum is quantized, Eq. (3), is highly arbitrary and introduced merely on the basis that it leads to results which agree with observation. It is now known that the angular momentum assignments in Eq. (3) are incorrect; that, for example, the ground state of hydrogen has zero angular momentum, not the one $\hbar$ of angular momentum assumed in the Bohr model. This is related to the fact that the ground state of hydrogen has spherical symmetry, not the planar symmetry implied by the Bohr model.

[4] See "Nuclear Magnetism and Hyperfine Structure" (MISN-0-541).

2. The well-established result of classical physics that an accelerating electric charge is the source of electromagnetic radiation is completely ignored in the Bohr model. While the electron is in stable orbit, it continuously experiences centripetal acceleration. The Bohr model assumes the electron does not radiate energy for that would lead to the rapid collapse of the atom with the electron quickly spiralling into the nucleus while emitting a continuum of frequencies. Radiation is assumed to be emitted only while the electron jumps from one stable orbit to another.

3. The actual mechanism of the electron jump is unspecified, and it is not clear what, if anything, triggers the jump and how one can describe the system at times when the electron is in transit from one stable orbit to another. Lacking such a description, it is impossible to account for the observed intensities of the spectral emission frequencies.

4. The Bohr-Sommerfeld approach, on extension to atomic systems with more than one electron, is incapable of giving correct results even after the much increased computational complexities are surmounted. This is, of course, a major limitation and was a source of considerable disappointment at the time of the introduction of the model.

5. It is just about impossible to see how atoms with pellet-like orbiting electrons can ever produce stable chemical combinations of any kind: the Bohr-Sommerfeld model does seem capable of producing an acceptable theory of the chemical bond.

## 8. The Quantum Mechanical Atom

**8a. Overview.** All the above difficulties were eventually removed through the discovery of the wave-like properties of electrons and the development of non-relativistic and, later, relativistic quantum mechanics as the theoretical framework for dealing with the fundamentally dual wave and particle character of the electron.

These theories de-emphasize the precise location and speed of the electron as a particle and encompass more natural but more elaborate versions of angular momentum quantization.[5]

[5]See "De Broglie Waves" (MISN-0-240).





**8b. Energy Levels.** For one-electron systems, Bohr's energy formula with Sommerfeld's corrections, is duplicated by modern quantum theory, although Sommerfeld's relativistic correction now has a different origin, namely, as a magnetic interaction between the spin of the electron on its own axis and its orbital motion, giving rise to the fine structure effects.

For systems with more than one electron, the quantum mechanical calculation of the allowed energies of the systems are lengthy and complex and are generally carried out numerically on high-speed computers. The most important point about these calculations is that the results agree with the experimental values to the accuracy of the computer algorithms, generally up to five significant digits.

Simple approximate energy formulas exist only for one-electron atoms and for the alkali metal atoms which are one-valence-electron atoms.

**8c. Spectral Line Widths.** One important prediction of the quantum theory is that atomic energy levels, except for the ground state level, are not infinitely sharp but show small, non-zero spreads in energy $\Delta E$ around their respective central values. As a result, the spectral emission frequencies also show a spread or "natural line width," in agreement with observation.[6] Other effects, including collisions and Doppler shifts, give spectral line broadening beyond that due to the natural line width.[7]

**8d. Hyperfine Structure.** Many nuclear isotopes have an intrinsic spin motion, the nuclear spin, which interacts magnetically with the electronic motion to produce so-called "hyperfine structure" effects which, along with other nuclear interaction effects, lead to a further improvement in agreement between theory and experiment (to about seven significant digits).[8] Still further improvement is obtained when quantization of the internal electric and magnetic fields of the atom are included. These quantum electrodynamical effects bring agreement between theory and experiment to eight or nine significant digits, the limit of accuracy of current measurement techniques.

## Acknowledgments

Preparation of this module was supported in part by the National Science Foundation, Division of Science Education Development and

[6]See "The Uncertainty Relations" (MISN-0-241).
[7]See "Transitions and Spectral Analysis" (MISN-0-216).
[8]See "Nuclear Magnetism and Hyperfine Structure" (MISN-0-541).

Research, through Grant #SED 74-20088 to Michigan State University.

## Glossary

- **Bohr radius**: the radius of the first Bohr orbit, often used as a unit of measurement on the atomic scale.
- **Rydberg constant**: a combination of fundamental physical constants that gives the magnitude of the hydrogen atom ground state energy (13.6 eV). The Rydberg constant can be derived from the Bohr Model.





## PROBLEM SUPPLEMENT

$hc = 1.2397\,\text{KeV nm}$

$c = 3 \times 10^8\,\text{m/s}$

amu$= 0.931\,\text{GeV}/c^2 = 931\,\text{MeV}/c^2$

$M_H = 1.00797\,\text{amu}$

1. If an electron behaved like a classical charged particle undergoing an acceleration (centripetal), it would radiate electromagnetic energy. Thus it would lose energy and subsequently begin to spiral downward and eventually collapse onto the nucleus. For a particle of charge $e$ the rate of radiation of energy is given by:

$$\text{power} = \frac{2}{3}\frac{m}{c}\frac{v^4}{r},$$

   where $m$ is the particle's mass, $v$ is the particle's speed, and $r$ is the orbital radius.[9] Calculate the length of time required for an electron in orbit at the first Bohr radius ($n = 1$) about a hydrogen nucleus ($Z = 1$) to radiate an amount of energy equivalent to its initial kinetic energy, causing the atom to collapse. Assume the radiated power remains constant.

2. A muon is a particle similar to an electron, with charge $-e$, but with a mass of $105.66\,\text{MeV}/c^2$, approximately 200 times the mass of an electron. A "$\mu$-mesic atom" can be constructed out of a proton and a muon, interacting electrostatically. Calculate the first three allowed energy levels of the $\mu$-mesic atom and draw an energy level diagram to illustrate them.

[9] W. Panofsky and M. Phillips, *Classical Electricity and Magnetism*, p. 365, Addison-Wesley (1962).

**Brief Answers**:

1. $t = 2.48 \times 10^{-15}$ sec

2. **Energy**

| Energy | | |
|---|---|---|
| zero | ——— | $n = \infty$ |
| $-312.4$ eV | ——— | $n = 3$ |
| $-703.0$ eV | ——— | $n = 2$ |
| $-2812$ eV | ——— | $n = 1$ |





# MODEL EXAM

1. See Output Skills K1-K5 in this module's *ID Sheet*. One or more of these skills, or none, may be on th actual exam.

2. Calculate the first four allowed energy states of a $C^{+5}$ hydrogen-like ion. Sketch the energy levels roughly to scale on an energy level diagram. The ground state energy of hydrogen is $-13.6$ eV. The normal carbon has 6 electrons.

**Brief Answers**:

1. See this module's *text*.

2. **Energy**

| Energy | | |
|---|---|---|
| zero | ——— | $n = \infty$ |
| $-30.6$ eV | ——— | $n = 4$ |
| $-54.4$ eV | ——— | $n = 3$ |
| $-122$ eV | ——— | $n = 2$ |
| $-490$ eV | ——— | $n = 1$ |